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Quantization of heterotic strings in a Gödel/anti de Sitter spacetime and chronology protection*

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Abstract: We show that a Gödel-like deformation of AdS_3 in heterotic string theory can be realized as an exact string background. Indeed this class of solutions is obtained as an exactly marginal deformation of the worldsheet conformal field theory describing the NS5/F1 heterotic background. It can also be embedded in type-II superstrings by Kaluza-Klein reduction. We compute the spectrum of this model as well as the genus one modular invariant partition function. We discuss the issue of closed timelike curves and the propagation of long strings. They destabilize completely the background, although we construct another exact string background that may describe the result of the condensation of these long strings. Closed timelike curves are avoided in that case.

Keywords: Superstring Vacua, Conformal Field Models in String Theory.

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1. Introduction

The fate of closed timelike curves (CTC's) in general relativity is a long standing problem. There is in particular a conjecture by Hawking [1] — the "Chronology Protection Conjecture" — which states that, in quantum gravity, a causally safe background cannot develop closed timelikes curves. One of the oldest known nontrivial spacetime with this pathology is the Gödel universe [2]. Spacetimes of this kind have been recently embedded in various supergravity theories [3, 4, 5, 6], in the hope that string/M theory would cast a new light on the fate of closed timelike curves in these spacetimes. All these works are based on an analysis of the low energy effective action and therefore do not take into account the full quantum string effects. Nevertheless some interesting observations were made while considering extended BPS objects (branes and supertubes) as probes in these spacetimes, which may destabilize the background [7]. Besides one type of Gödel universe is T-dual to an exact type-IIA pp-waves background. This opened the door for exact quantization of string probes [8, 9, 6, 10], albeit only in the lighthcone gauge.

In this note we would like to point out that a one parameter class of spacetimes interpolating between AdS_3 and the Gödel universe constructed in [11] — and recently discussed in [12] in the context of general relativity — can actually be embedded as an *exact heterotic string background*. It turns out that it is obtained as an asymmetric deformation of the $\mathrm{SL}(2,\mathbb{R})_L \times \mathrm{SL}(2,\mathbb{R})_R$ conformal field theory (all the left-right symmetric deformations have been considered in a previous paper [13]) and of the current algebra of the gauge group. If we want to consider type-II superstrings — and in that case the background will be supersymmetric — we can choose an internal compact U(1) instead. The CFT description of this background allows us to compute the exact string spectrum and partition function. We can also trace back the different kind of states which appear in the spectrum in terms of $\mathrm{SL}(2,\mathbb{R})$ representations, and show that the no-ghost theorem for $\mathrm{SL}(2,\mathbb{R})$ can be extended to the deformed theory.

Then we discuss long string states to see if something special happens due to the closed timelike curves around which they can wind. We will show explicitly that the long strings seems to destabilize the background, because their spectrum becomes highly tachyonic. We propose a mechanism to solve the causality problem, by the condensation of a ring of fundamental strings, akin to the enhancon mechanism [14]. The endpoint of this condensation can also be realized as an exact conformal field theory, which does not contain instabilities as we shall see. This can be interpreted as a stringy chronology protection.

The paper is organized as follows. In section 2, we study the sigma model corresponding to the Gödel/AdS_3 solution, i.e. the solution for the background fields. In section 3 we compute the spectrum for all the string excitations in this background, and then the modular invariant partition function at genus one. In section 4 we consider the long string solutions, classically and at the level of the exact string background. We propose a scenario in section 5 to solve the instability by condensing fundamental strings. Finally in section 6 we summarize the main results of the paper and discuss holography. Appendix A is a short review of the spectrum and partition function for AdS_3; in appendix B we give several coordinate systems on the Gödel/AdS_3 spacetime. An outline of the proof of the no-ghost theorem is given in appendix C.

2. The Gödel/AdS_3 solution

2.1 The worldsheet conformal field theory

We start with the $AdS_3 \times S^3 \times K3$ supersymmetric background of heterotic string theory. It represents a configuration of wrapped NS5-branes and fundamental strings [15, 16, 17].

The part of the worldsheet action which is relevant to discuss the deformation comprises the $\mathrm{SL}(2,\mathbb{R})_k$ — AdS_3 — part and the worldsheet right-moving current algebra:

$$S = \frac{1}{4\pi}\int d^2z\, \frac{k}{2}\left(\partial r\bar{\partial} r - \partial\alpha\bar{\partial}\alpha - \partial\beta\bar{\partial}\beta - 2\cosh r\,\partial\alpha\bar{\partial}\beta\right) + \eta_{\mu\nu}\psi^\mu\bar{\partial}\psi^\nu + \lambda^I\partial\lambda^I \tag{2.1}$$

Here the ψ^μ are the worldsheet fermions of the holomorphic CFT (supersymmetric side), while the antiholomorphic fermions λ^I are the internal fermions of the gauge sector, with currents: $\bar{J}^{IJ} = i : \lambda^I(\bar{z})\lambda^J(\bar{z}) :$. We would like to turn on a gauge field by switching on the operator:

$$V(z,\bar{z}) = \left(J^3 + i\psi^+\psi^-\right)\bar{J}^a\,, \tag{2.2}$$

where

$$J^3 = k\left(\partial\beta + \cosh r\,\partial\alpha\right). \tag{2.3}$$

Such a deformation is not possible around flat space because the corresponding operator:

$$V_{\text{flat}} = F^a_{ij}\left(\frac{1}{2}x^i\partial x^j + \psi^i\psi^j\right)\bar{J}^a$$

is not marginal. However a magnetic deformation of the NS5-branes background has been studied some years ago [18]. It was first introduced in the context of infrared regularization of superstrings [19].[1]

Our treatment of AdS_3 deformation will be close in spirit. Switching on this gauge field in the bulk, and taking into account the back-reaction on the metric and the other fields, we obtain the solution of the sigma-model equations:

$$\begin{aligned}
\frac{4}{k}ds^2 &= dr^2 - (1+2H^2\cosh^2 r)d\alpha^2 - (1+2H^2)d\beta^2 - 2(1+2H^2)\cosh r\,d\alpha d\beta \\
B &= \frac{k}{4}\cosh r\,d\beta\wedge d\alpha \\
A &= -g\sqrt{k}H\left[\cosh r\,d\alpha + d\beta\right].
\end{aligned} \tag{2.4}$$

The metric can be also rewritten as a squashed AdS_3 geometry, as was first noticed in [20]:

$$ds^2 = \frac{k}{4}\left[dr^2 + \sinh^2 r\,d\alpha^2 - (1+2H^2)\left(d\beta + \cosh r\,d\alpha\right)^2\right]. \tag{2.5}$$

The propagation of scalar fields in this spacetime has been recently addressed in [12]. The original Gödel spacetime is obtained for the special value of the deformation parameter:

$$H_{\text{Gödel}} = \frac{1}{\sqrt{2}}\,.$$

The relation with AdS_3 in global coordinates becomes more obvious after the change of coordinates:

$$\alpha = t+\phi\,,\qquad \beta = t-\phi\,,\qquad r = 2\rho\,.$$

[1]Other magnetic backgrounds for the heterotic string have been constructed in [9].

Then we obtain the solution:

$$\begin{aligned}
\frac{1}{k}ds^2 &= d\rho^2 - \cosh^2\rho\, dt^2 + \sinh^2\rho\, d\phi^2 - 2H^2\left(\cosh^2\rho\, dt + \sinh^2\rho\, d\phi\right)^2 \\
B &= \frac{k}{2}\cosh 2\rho\, d\phi \wedge dt \\
F &= dA = g\sqrt{k}H \sinh 2\rho\, (dt + d\phi) \wedge d\rho\,.
\end{aligned} \tag{2.6}$$

By construction, the isometry group of this spacetime is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1)$.

2.2 Causal structure

This section is a short review of the results of [20, 12] that we give for completeness. The striking feature of the class of backgrounds (2.5) is that there are closed timelike curves spanned by the coordinate ϕ above the critical radius:

$$G_{\phi\phi} = k\sinh^2\rho_c(1 - 2H^2\sinh^2\rho_c) = 0 \;\rightarrow\; \rho_c = \operatorname{arcsinh}\frac{1}{\sqrt{2}H}\,. \tag{2.7}$$

These closed timelike curves are contractible, hence the spacetime is topologically trivial. Since the spacetime is homogeneous, they are closed timelike curves going through each point of the manifold. Although translation in the coordinate t corresponds to a globally defined timelike Killing vector, we cannot globally define a time function because the sections of the spacetime at constant t are not spacelike hypersurfaces, due to the closed timelike curves. The Cauchy problem is not well-defined in this spacetime.

The analysis of the geodesics in these backgrounds has been made in hyperbolic coordinates, see appendix B. The authors of [12] observed that the null and timelike geodesics projects onto full circles of arbitrary center. For the spacelike geodesics it is true only for those with $\rho > \rho_c$, i.e. those projecting onto closed timelike curves. In AdS_3, it has been shown [23] that the action of spectral flow on spacelike geodesics produces long strings that can reach the boundary of anti de Sitter and wind around the ϕ-coordinate; in the Gödel-deformed background, as we will see, these long string states are still present, and can wind around the closed timelike curves, causing a potential instability.

2.3 Conformal invariance

We now have to show that the deformation of the conformal field theory is truly marginal to all orders in the deformation. We bosonize the right-moving gauge current used for the deformation as: $\bar{J}^a = i\bar{\partial}Y$. We will work in bosonic strings for simplicity; in that case $\bar{J}^a$ has a left-moving partner. Then $Y(z,\bar{z})$ can be viewed as an internal coordinate, and the background will be obtained by a Kaluza-Klein reduction. We write the relevant part of the sigma-model for the deformed geometry in the Kaluza-Klein form as:

$$\begin{aligned}
S_{\text{def}} = \frac{1}{4\pi}\int d^2z\,\frac{k}{2}\Big\{&\partial r\bar{\partial}r - \partial\alpha\bar{\partial}\alpha - \partial\beta\bar{\partial}\beta - 2\cosh r\partial\alpha\bar{\partial}\beta - \\
&- 2H^2\left[\cosh^2 r\partial\alpha\bar{\partial}\alpha + \partial\beta\bar{\partial}\beta + \cosh r\left(\partial\alpha\bar{\partial}\beta + \partial\beta\bar{\partial}\alpha\right)\right]\Big\} + \\
&+ \left[\partial Y - \sqrt{k}H\left(\cosh r\partial\alpha + \partial\beta\right)\right]\left[\bar{\partial}Y - \sqrt{k}H\left(\cosh r\bar{\partial}\alpha + \bar{\partial}\beta\right)\right].
\end{aligned} \tag{2.8}$$

To prove conformal invariance, we note, as in [18] that one may rewrite the action as:

$$
\begin{aligned}
S_{\mathrm{SL}(2,\mathbb{R})}(r,\alpha,\beta) + \frac{1}{4\pi}\int d^2z\, \partial Y \bar{\partial} Y - \frac{\sqrt{k}H}{2\pi}\int d^2z\,(\partial\beta + \cosh r\,\partial\alpha)\,\bar{\partial}Y = \\
= S_{\mathrm{SL}(2,\mathbb{R})}\left(r,\alpha,\beta + \frac{2H}{\sqrt{k}}Y\right) + \frac{1+2H^2}{4\pi}\int d^2z\,\partial Y\bar{\partial}Y
\end{aligned}
\qquad (2.9)
$$

where we have dropped out a total derivative:

$$
2\sqrt{k}H\int d^2z\left(\partial Y\bar{\partial}\beta - \bar{\partial}Y\partial\beta\right);
$$

this term will however give rise to topological contributions to the spectrum. So this background is an exactly conformal invariant sigma-model, to all orders in α'. It turns out that the last expression (2.9) is exactly the same as the sigma-model of the deformed background, eq. (2.8). In section 3 we will compute explicitely the spectrum for the heterotic background. We get also from eq. (2.9) the modified expression for the J^3 current:

$$
J^3_d = k\,(\partial\beta + \cosh r\partial\alpha) + 2H\sqrt{k}\partial Y\,, \qquad (2.10)
$$

and the $\mathrm{SL}(2,\mathbb{R})_L$ symmetry is broken down to U(1). Of course, the $\mathrm{SL}(2,\mathbb{R})_R$ is preserved by the deformation.

2.4 Target-space supersymmetry

The supersymmetry variations for the gaugino, the dilatino and the gravitino are:

$$
\begin{aligned}
\delta\chi &= F_{ab}\gamma^{ab}\,\epsilon \\
\delta\lambda &= \left(\gamma^a\partial_a\phi - \frac{1}{6}H_{abc}\gamma^{abc}\right)\epsilon \\
\delta\Psi_a &= \left[\partial_a + \frac{1}{4}\left(\omega_a^{bc} - H_a^{bc}\right)\gamma_{ab}\right]\epsilon\,.
\end{aligned}
\qquad (2.11)
$$

From the dilatino variation we obtain:

$$
\left[\Gamma^3\Gamma^4\Gamma^5 - \frac{1}{\sqrt{1+2H^2}}\Gamma^0\Gamma^1\Gamma^2\right]\epsilon = 0 \qquad (2.12)
$$

where 0,1,2 are the tangent space indices for AdS_3 and 3,4,5 those of S^3. Thus, the supersymmetry is completely broken by the deformation. The analysis of the gaugino and the gravitino equations would lead to the same conclusion. However for the consistency of the string theory one still has to perform the GSO projection; the possible appearance of tachyonic modes will be addressed in section 4. If this solution is embedded in type-II superstrings, as a Kaluza-Klein reduction — see eq. (2.8) — the background is a supersymmetric solution of the corresponding Kaluza-Klein supergravity, as in [21]. It will be shown in section 3.

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3. The string spectrum and one-loop partition function

The partition function for type-IIB superstring theory on $AdS_3 \times S^3 \times K3$ was obtained in [13]. The extension to heterotic case is straightforward, by using the heterotic map [22]. For concreteness, we consider an $E_8 \times E_8 \to E_8 \times E_7 \times \mathrm{SU}(2) \times \mathrm{U}(1)^4$ compactification:

$$\begin{aligned} Z(\tau) = & \frac{\mathrm{Im}\,\tau}{\eta^2 \bar\eta^{14}} Z_{\mathrm{SU}(2)} Z_{\mathrm{SL}(2,\mathbb{R})} \frac{1}{2} \sum_{h,g=0}^{1} Z_{4,4}\begin{bmatrix} h \\ g \end{bmatrix} \frac{1}{2} \sum_{a,b=0}^{1} (-)^{a+b} \vartheta^2 \begin{bmatrix} a \\ b \end{bmatrix} \vartheta \begin{bmatrix} a+h \\ b+g \end{bmatrix} \vartheta \begin{bmatrix} a-h \\ b-g \end{bmatrix} \\ & \times \bar\Gamma_{E_8} \times \frac{1}{2} \sum_{\gamma,\delta=0}^{1} \bar\vartheta \begin{bmatrix} \gamma+h \\ \delta+g \end{bmatrix} \bar\vartheta \begin{bmatrix} \gamma-h \\ \delta-g \end{bmatrix} \bar\vartheta^6 \begin{bmatrix} \gamma \\ \delta \end{bmatrix} . \end{aligned} \tag{3.1}$$

The twisted torus conformal blocks are:

$$Z_{4,4}\begin{bmatrix} h \\ g \end{bmatrix} = \frac{\Gamma_{4,4}}{\eta^4 \bar\eta^4} \quad \text{for} \quad h = g = 0\,, \qquad \text{and} \qquad Z_{4,4}\begin{bmatrix} h \\ g \end{bmatrix} = \frac{16 \eta^2 \bar\eta^2}{\left| \vartheta \left[{1-h \atop 1-g} \right] (0|\tau) \right|^4} \quad \text{otherwise.} \tag{3.2}$$

The $\mathrm{SU}(2)_k$ bosonic conformal block is chosen to be the standard diagonal invariant:

$$Z_{\mathrm{SU}(2)} = \sum_{l=0}^{k} \chi_k^l \bar\chi_k^l \,, \tag{3.3}$$

and the $\mathrm{SL}(2,\mathbb{R})_{k+4}$ partition function is given in appendix A.

3.1 Exact string spectrum in the Gödel/AdS spacetime

Now we would like to perform the marginal deformation (2.2) in this CFT heterotic background. As it is known generically for conformal field theories based on affine algebras, the deformation will act on the weight lattice of the subgroup along which the deformation is performed (see [24] for a recent discussion):

- The abelian *elliptic subgroup* of the supersymmetric $\mathrm{SL}(2,\mathbb{R})_{k+2}$, generated by $\mathcal{J}^3 = J^3 + i\psi^+\psi^-$, for the left-movers,
- a $\mathrm{U}(1)$ subgroup of the gauge group for the right-movers.

The left and right weights of the relevant lattices are:

$$\begin{aligned} L_0 &= -\frac{1}{k+4}\left(\tilde m + \frac{k+4}{2} w_+ \right)^2 + \frac{1}{2}\left(n + \frac{a}{2} \right)^2 \\ &= -\frac{1}{k+2}\left(\tilde m + \frac{k+4}{2} w_+ + n + \frac{a}{2} \right)^2 + \frac{k+4}{2(k+2)}\left(n + \frac{a}{2} + w_+ + \frac{2\tilde m}{k+4} \right)^2 \end{aligned} \tag{3.4}$$

$$\bar L_0 = -\frac{1}{k+4}\left(\bar{\tilde m} + \frac{k+4}{2} w_+ \right)^2 + \frac{1}{2}\left(\bar n + \frac{\gamma}{2} \right)^2 . \tag{3.5}$$

Here $n + a/2$ corresponds to the charges of the left fermionic current $\psi^+\psi^-$, and $a = 0, 1$ are respectively the NS and the R sector, as in the partition function. Also $\bar n + \gamma/2$ corresponds to the weights lattice of the gauge current which is picked out by the deformation. In the expression for L_0, eq. (3.4), we have explicitely factorized the supersymmetric weights of $\mathrm{SL}(2,\mathbb{R})_{k+2}$. Then the deformation acts as a $O(2)$ rotation between these weights and the

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right weights of the U(1) gauge field,[2] parameterized by the angle ζ; we obtain then:

$$
\begin{aligned}
L_0^{\text{def}} &= -\left[\frac{\cos\zeta}{\sqrt{k+2}}\left(\tilde{m}+\frac{k+4}{2}w_+ + n + \frac{a}{2}\right)+\frac{\sin\zeta}{\sqrt{2}}\left(\bar{n}+\frac{\gamma}{2}\right)\right]^2 + \\
&\quad +\frac{k+4}{2(k+2)}\left(n+\frac{a}{2}+w_+ + \frac{2\tilde{m}}{k+4}\right)^2
\end{aligned}
\tag{3.6}
$$

$$
\begin{aligned}
\bar{L}_0^{\text{def}} &= -\frac{1}{k+4}\left(\bar{\tilde{m}}+\frac{k+4}{2}w_+\right)^2 + \\
&\quad +\left[\frac{\cos\zeta}{\sqrt{2}}\left(\bar{n}+\frac{\gamma}{2}\right)-\frac{\sin\zeta}{\sqrt{k+2}}\left(\tilde{m}+\frac{k+4}{2}w_+ + n + \frac{a}{2}\right)\right]^2
\end{aligned}
\tag{3.7}
$$

where the relation between this parameterization and the vev of the gauge field is:

$$
\cos^2\zeta = \frac{1}{1+2H^2}\,. \tag{3.8}
$$

Note that there is no maximal value for the gauge field, unlike the magnetic NS5-brane background [18].

In the type-II superstrings (Kaluza-Klein reduction), we replace the lattice from the right-moving heterotic gauge sector by the right lattice of a compact internal coordinate. In this case it is obvious that the deformation preserves 8 supercharges coming from the right-moving fermionic sector, left untouched by the deformation.

3.2 One-loop vacuum amplitude

To compute the partition function for heterotic strings on Gödel/$\mathrm{AdS}_3 \times S^3 \times K3$ one simply has to combine the previous analysis of the spectrum with the remaining ingredients. Then the partition function of the deformed model reads (see appendix A):

$$
\begin{aligned}
Z_{\text{Gödel}}(\tau) &= \frac{\operatorname{Im}\tau}{\eta^3\bar{\eta}^{15}} Z_{\mathrm{SU}(2)} \frac{1}{2}\sum_{h,g=0}^{1} Z_{4,4}\begin{bmatrix}h\\g\end{bmatrix} \times \\
&\quad \times \frac{1}{2}\sum_{a,b=0}^{1}(-)^{a+b}\,\vartheta\begin{bmatrix}a\\b\end{bmatrix}\vartheta\begin{bmatrix}a+h\\b+g\end{bmatrix}\vartheta\begin{bmatrix}a-h\\b-g\end{bmatrix} \times \\
&\quad \times \bar{\Gamma}_{E_8} \times \frac{1}{2}\sum_{\gamma,\delta=0}^{1}\bar{\vartheta}\begin{bmatrix}\gamma+h\\\delta+g\end{bmatrix}\bar{\vartheta}\begin{bmatrix}\gamma-h\\\delta-g\end{bmatrix}\bar{\vartheta}^5\begin{bmatrix}\gamma\\\delta\end{bmatrix} \times \\
&\quad \times \int d^2t\, Z_{\text{cigar}}\begin{bmatrix}-t_1\\-t_2\end{bmatrix} \sum_{N,W,n,\bar{n}\in\mathbb{Z}} e^{i\pi\left(2Nt_2+b(n+\frac{a}{2})-\delta(\bar{n}+\frac{\gamma}{2})\right)} \times \\
&\quad \times q^{-\left[\frac{\cos\zeta}{\sqrt{k+2}}\left(\frac{N}{2}+\frac{k+4}{2}(W+t_1)+n+\frac{a}{2}\right)+\frac{\sin\zeta}{\sqrt{2}}\left(\bar{n}+\frac{\gamma}{2}\right)\right]^2+\frac{k+4}{2(k+2)}\left(n+\frac{a}{2}+(W+t_1)+\frac{2N}{k+4}\right)^2} \times \\
&\quad \times \bar{q}^{-\frac{1}{k+4}\left(\frac{N}{2}-\frac{k+4}{2}(W+t_1)\right)^2+\left[\frac{\cos\zeta}{\sqrt{2}}\left(\bar{n}+\frac{\gamma}{2}\right)-\frac{\sin\zeta}{\sqrt{k+2}}\left(N+\frac{k+4}{2}(W+t_1)+n+\frac{a}{2}\right)\right]^2}\,.
\end{aligned}
\tag{3.9}
$$

[2]It is of course the analytic continuation of the usual $O(1,1)$ transformation of toroidal lattices, but here the lattice of J^3 is timelike.

This partition function is modular invariant by construction since it is obtained as an $O(2, 2, \mathbb{R})$ rotation of an even self-dual lattice. Due to the mixing between the fermionic characters and the lattice of J^3, it is not possible to use a Jacobi identity to prove target space supersymmetry; all the supersymmetry is broken in this background, as expected from the target space analysis.

4. The fate of long strings

The long strings, corresponding to the continuous representations of $\mathrm{SL}(2, \mathbb{R})$ are the natural probes for the closed timelike curves since they can wind around the coordinate ϕ and become macroscopic. Such long strings in Gödel universe T-dual to pp-waves have been considered recently in [10], both from the classical and quantum point of view in a light-cone gauge appropriate to this problem. No tachyons were found in that case, although the authors pointed out a possibility of unitarity violation. Another analysis for M2-branes in M-theory Gödel universes has been made in [25].

4.1 Classical analysis

We would like to understand the behavior of simple long strings solutions. We will consider classical solutions to the Nambu-Goto action:

$$S = -T_1 \int d^2\xi \sqrt{G} + T_1 \int B_{[2]} \tag{4.1}$$

4.1.1 Long strings in AdS_3

In the undeformed AdS_3 geometry, we parameterize a long string wrapping w times the ϕ coordinate, without angular momentum, as:

$$\begin{cases} t = w_+ \tau \\ \phi = w_+ \sigma \\ \rho = \rho(\tau) \end{cases} \tag{4.2}$$

where we have used the static gauge for the time coordinate. So the classical Nambu-Goto action for a F1-string probe is:

$$S = -\pi k w_+ T_1 \int d\tau \left(\sinh\rho \sqrt{w_+^2 \cosh^2\rho - \dot{\rho}^2} - w_+ \cosh^2\rho \right) \tag{4.3}$$

with the equation of motion:

$$-\frac{d}{d\tau} \frac{\dot{\rho}\sinh\rho}{\sqrt{w_+^2 \cosh^2\rho - \dot{\rho}^2}} = \frac{w_+^2 \cosh 2\rho - \dot{\rho}^2}{\sqrt{w_+^2 \cosh^2\rho - \dot{\rho}^2}} \cosh\rho - w_+ \sinh 2\rho \, .$$

We have a long string solution:

$$\rho = w_+ |\tau| \, ;$$

so the long string reach the boundary of AdS_3 for $t = \pm\infty$. Qualitatively this behavior appears because the attractive force due to the string tension and the repulsive force due to the NS-NS two-form cancel. This simple solution corresponds to the spectral flow of a spacelike geodesic. More general solutions have being obtained using a group manifold approach in [23].

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4.1.2 Long strings in AdS_3/Gödel

Now we would like to examine long string solutions in the AdS/Gödel space. The crucial feature is to check if such long strings can reach the critical radius and then wrap closed timelike curves. We consider classical macroscopic strings which are uncharged under the gauge group for simplicity. As in the previous case, we are looking for solutions winding around ϕ, without angular momentum, eq. (4.2). Then the Nambu-Goto action is given by:

$$S = -\pi k w_+ T_1 \int d\tau \left(\sinh\rho \sqrt{(1+2H^2) w_+^2 \cosh^2\rho - (1-2H^2\sinh^2\rho)\dot{\rho}^2} - w_+ \cosh^2\rho \right) . \tag{4.4}$$

So we obtain the equation of motion for $\rho(\tau)$:

$$\begin{aligned} -\frac{d}{d\tau} & \frac{(1-2H^2\sinh^2\rho)\sinh\rho\,\dot{\rho}}{\sqrt{(1+2H^2)w_+^2\cosh^2\rho - (1-2H^2\sinh^2\rho)\dot{\rho}^2}} = \\ = & \frac{(1+2H^2)w_+^2\cosh\rho(\cosh^2\rho + \sinh^2\rho) + (4H^2\sinh^2\rho - 1)\cosh\rho\dot{\rho}^2}{\sqrt{(1+2H^2)w_+^2\cosh^2\rho - (1-2H^2\sinh^2\rho)\dot{\rho}^2}} \cosh\rho - w_+ \sinh 2\rho . \end{aligned}$$

We still have also long string solutions with:

$$\rho(\tau) = \sqrt{1+2H^2}\; w_+ |\tau| .$$

The radius of the worldsheet of these long strings grows linearly with time, until they eventually wrap the closed timelike curves. We conclude that the Gödel deformation changes the on-shell values of the momenta, but otherwise leads to the same qualitative behavior of the long string solutions as in AdS_3: long strings can escape to infinity with a finite energy.

4.2 Unitarity of the physical spectrum

Naively, we would expect that the string spectrum would contain ghosts for long string solutions. In fact, the kinetic term for the field $\phi(z,\bar{z})$ in the Polyakov action is:

$$\mathcal{L} \sim k \sinh^2 r (1 - 2H^2 \sinh^2 r) \partial\phi\bar{\partial}\phi ,$$

so, for $r > r_c$ it seems that the sign of the kinetic term for the fluctuations of ϕ become negative. But the off-diagonal term between $\phi(z,\bar{z})$ and $t(z,\bar{z})$ makes the analysis more subtle; indeed, the determinant of the metric of the deformed spacetime is:

$$-g = (1+2H^2)\left(\frac{k}{4}\right)^3 \sinh^2 r ,$$

which is always of the same sign.

To have a precise answer to this crucial issue, we will adapt the no-ghost theorem for $SL(2,\mathbb{R})$, to the case of the Gödel-deformed $SL(2,\mathbb{R})$. We would like to see if the modification of the structure of the zero modes changes the conclusion about the suppression of negative-norm states by the Virasoro conditions. A outline of the no-ghost theorem proof is given in appendix C. The conclusion is: *the spectrum doesn't contain negative-norm states.*

4.3 Long strings spectrum

Now we consider the exact string spectrum. We would like to see if the spectrum contains tachyons, i.e. on-shell states with an imaginary value for the energy:[3]

$$E = \mathcal{J}_0^3 + \bar{J}_0^3 = \tilde{m} + \bar{\tilde{m}} + n + \frac{a}{2} + (k+2)w_+ \, . \tag{4.5}$$

The mass-shell condition for the continuous representations reads (*"other"* indicates the internal CFT):

$$\begin{aligned} L_0 &= \frac{s^2+1/4}{k+2} - w_+ \left(\tilde{m} + \tilde{n} + \frac{a}{2}\right) - \frac{k+2}{4} w_+^2 + \frac{1}{2}\left(\tilde{n} + \frac{a}{2}\right)^2 + \\ &\quad + \frac{\sin\zeta}{k+2} \left\{ \left[\left(\tilde{m} + \frac{k+2}{2} w_+ + \tilde{n} + \frac{a}{2}\right)^2 - \frac{k+2}{2}\left(\bar{n} + \frac{\gamma}{2}\right)^2 \right] \sin\zeta - \right. \\ &\qquad \left. - \sqrt{2(k+2)} \left(\tilde{m} + \frac{k+2}{2} w_+ + \tilde{n} + \frac{a}{2}\right)\left(\bar{n} + \frac{\gamma}{2}\right) \cos\zeta \right\} + N + h_{\text{other}} - \frac{1}{2} \\ &= 0 \end{aligned} \tag{4.6}$$

$$\begin{aligned} \bar{L}_0 &= \frac{s^2+1/4}{k+2} - w_+ \bar{\tilde{m}} - \frac{k+4}{4} w_+^2 + \frac{1}{2}\left(\bar{n} + \frac{\gamma}{2}\right)^2 + \\ &\quad + \frac{\sin\zeta}{k+2} \left\{ \left[\left(\tilde{m} + \frac{k+2}{2} w_+ + \tilde{n} + \frac{a}{2}\right)^2 - \frac{k+2}{2}\left(\bar{n} + \frac{\gamma}{2}\right)^2 \right] \sin\zeta - \right. \\ &\qquad \left. - \sqrt{2(k+2)} \left(\tilde{m} + \frac{k+2}{2} w_+ + \tilde{n} + \frac{a}{2}\right)\left(\bar{n} + \frac{\gamma}{2}\right) \cos\zeta \right\} + \bar{N} + \bar{h}_{\text{other}} - 1 \\ &= 0 \, . \end{aligned} \tag{4.7}$$

In a supersymmetric WZW model, the spectral flow must act also on the fermions, so $n = \tilde{n} - w_+$ in the w_+-flowed sector (see [26]); it amounts to redefining the vacuum. We also have to act with the spectral flow in the gauge sector, in order to satisfy the matching condition $L_0 = \bar{L}_0$. At this point we have to decide if we act on the U(1) along which we switch on a field strength — *choice (i)* — or an other unbroken Cartan generator — *choice (ii)*. Since we have constructed the deformation such that it preserves $L_0 - \bar{L}_0$, both choices are equally acceptable. However the spectrum is slightly different. In both cases, the matching condition will be satisfied, provided that:

$$w_+ \left(\tilde{m} - \bar{\tilde{m}}\right) = w_+ \ell = h_{\text{other}} - \bar{h}_{\text{other}} \, ,$$

where ℓ is the angular momentum.

We take for example the "massless" state:

- left : $|s, p\rangle \otimes \psi^{\text{other}}_{-1/2} |0\rangle_{NS}$
- right: $\bar{\alpha}^{\text{other}}_{-1} |s, \bar{p}\rangle \otimes |0\rangle_{NS}$.

[3] As we have seen there is no globally defined notion of energy in this spacetime. At any rate, we will take the same definition as in AdS_3 — the conjugate to the t translations — since the corresponding isometry is unbroken by the deformation.

Then we have, with the embedding of the spectral flow *(ii):*

$$L_0^{(ii)} = \frac{s^2 + 1/4 + \tilde{m}^2 \sin^2 \zeta}{k+2} - \frac{k+2}{4} w_+^2 \cos^2 \zeta - w_+ \tilde{m} \cos^2 \zeta + \frac{1}{2} p^2 \,. \tag{4.8}$$

So we have the mass-shell condition for the state under consideration:

$$\frac{s^2 + 1/4}{k+2} + \frac{k+2}{4} w_+^2 - w_+ \frac{E}{2} + \frac{\sin^2 \zeta}{4(k+2)} E^2 + \frac{1}{2} p^2 = 0 \,. \tag{4.9}$$

And

$$E = \frac{k+2}{\sin^2 \zeta} \left(w_+ \pm \sqrt{w_+^2 \cos^2 \zeta - \frac{4 \sin^2 \zeta}{k+2} \left(\frac{s^2 + 1/4}{k+2} + \frac{1}{2} p^2 \right)} \right) . \tag{4.10}$$

If for example we increase the mass of the state with some momentum p in the compact internal CFT, the energy of the state will eventually become imaginary. This strange behavior is due to the quadratic term in the energy with a positive sign. The critical value is:

$$\frac{s^{(ii)\ 2} + 1/4}{k+2} + \frac{1}{2} p^2 = \frac{k+2}{4} w_+^2 \operatorname{cotan}^2 \zeta \,.$$

For a more general state with oscillators, the same statement holds. An infinite number of massive long string states will become tachyonic when the Gödel deformation is turned on (other tachyonic instabilities in magnetic fields have been discussed in [9]); these instabilities are likely due to the wrapping of closed timelike curves.

For the other embedding of spectral flow — *choice (i)* — we have instead:

$$\begin{aligned} L_0^{(i)} &= \frac{s^2 + 1/4 + \tilde{m}^2 \sin^2 \zeta}{k+2} - \frac{1}{2} \left(\sqrt{\frac{k+2}{2}} \cos \zeta + \sin \zeta \right)^2 w_+^2 - \\ &\quad - \cos \zeta \left(\cos \zeta + \sqrt{\frac{2}{k+2}} \sin \zeta \right) w_+ \tilde{m} + \frac{1}{2} p^2 \end{aligned} \tag{4.11}$$

and we find a correction to the critical value of momenta:

$$\frac{s^{(i)\ 2} + 1/4}{k+2} + \frac{1}{2} p^2 = \frac{k+2}{4} w_+^2 \left[\operatorname{cotan} \zeta + \sqrt{\frac{2}{k+2}} \right]^2 ,$$

but otherwise the conclusions are the same.

Behavior of short strings. Since the discrete representations correspond to strings trapped in the center of the spacetime (around $r = 0$), we expect that the corresponding spectrum is well-behaved. For the discrete representations, we have:

$$is = \frac{1}{2} - \tilde{j} \,,$$

and also a relation between the spin $\tilde{j}$ and the $\tilde{J}^3$ eigenvalue: $\tilde{m} = \tilde{j} + q$, with $q \in \mathbb{N}$, for the primaries of the lowest weight representations. The spin is also restricted for unitarity to

$$\frac{1}{2} < \tilde{j} < \frac{(k+3)}{2} \tag{4.12}$$

We can solve the mass-shell condition for $\tilde{j}$:

$$L_0^{\mathcal{D}^+} = -\frac{\tilde{j}(\tilde{j}-1)}{k+2} - \frac{k+2}{4} w_+^2 \cos^2\zeta - w_+(\tilde{j}+q)\cos^2\zeta + \frac{\sin^2\zeta}{k+2}(\tilde{j}+q)^2 + h_{\text{int}} = 0\,, \quad (4.13)$$

and find:

$$\begin{aligned}\tilde{j} = &\frac{1}{2\cos^2\zeta} - \frac{k+2}{2} w_+ + q\tan^2\zeta + \\ &+ \frac{1}{\cos^2\zeta}\sqrt{\frac{1}{4} + (k+2)\left(N_w + h_{\text{int}} - \frac{w_+}{2}\right)\cos^2\zeta + q(q+1)\sin^2\zeta}\,, \end{aligned} \quad (4.14)$$

where $N_w \equiv N - w_+ q$ is the level of the algebra after spectral flow. So we observe that the only effect of the Gödel deformation will be that the range of values for the spin allowed by the unitary bound (4.12) will impose different constraints on the internal weights. It would be interesting to relate this short string spectrum with the analysis of energy levels for a point particle in [25] and [12]. It was observed that this setup can be mapped onto a Landau problem.

5. Can gravitational deformation cure the pathology?

We have seen that the Gödel deformation of AdS_3, supersymmetric or not, completely destabilizes the background. Then we expect some kind of closed tachyon condensation; the description of this process is out of reach at present, but we can try to find the background that would correspond to the endpoint of the tachyon decay. In other words we are looking for another point in the moduli space of worldsheet conformal field theories which could have this interpretation.

5.1 Switching on more background fields

There is an additional deformation of the background which is exactly marginal, generated by the bilinear of the unbroken currents

$$V_g \sim (J^3 + i\psi^+\psi^-)(\bar{J}^3 - i\bar{\psi}^+\bar{\psi}^-)\,.$$

We would like to check if such a deformation can eliminate the closed timelike curves. This deformation have been studied in [13] around the $\mathrm{SL}(2,\mathbb{R})$ point. Here we would like to perform this deformation *in combination with Gödel deformation*. This is possible since each deformation preserves the $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry generating the other one. The solution of the sigma-model equations gives (see [18]):

$$\begin{aligned}\frac{4}{k} ds^2 = dr^2 &- \frac{(\lambda^2+1)^2 + (8H^2\lambda^2 - (\lambda^2-1)^2)\cosh^2 r}{\left(\lambda^2+1+(\lambda^2-1)\cosh r\right)^2} d\alpha^2 - \\ &- \frac{(\lambda^2+1)^2 + 8H^2\lambda^2 - (\lambda^2-1)^2\cosh^2 r}{\left(\lambda^2+1+(\lambda^2-1)\cosh r\right)^2} d\beta^2 - \\ &- 2\frac{4\lambda^2(1+2H^2)\cosh r - (\lambda^4-1)\sinh^2 r}{\left(\lambda^2+1+(\lambda^2-1)\cosh r\right)^2} d\alpha d\beta\end{aligned}$$

$$B = \frac{k}{4}\frac{\lambda^2 - 1 + (\lambda^2+1)\cosh r}{\lambda^2 + 1 + (\lambda^2-1)\cosh r} d\beta \wedge d\alpha$$
$$A = \frac{2g\sqrt{k}H\lambda}{\lambda^2 + 1 + (\lambda^2-1)\cosh r}\left(d\beta + \cosh r d\alpha\right)$$
$$e^{2\Phi} = \frac{\lambda e^{2\Phi_0}}{\lambda^2 + 1 + (\lambda^2-1)\cosh r}\,. \tag{5.1}$$

The model without $J^3\bar{J}^3$ deformation corresponds to $\lambda = 1$. The scalar curvature of this manifold is given by:

$$R(H,\lambda) = -\frac{8}{k}\frac{5\lambda^2 - 1 - \lambda^4 - 2H^2\lambda^2 + (1-\lambda^4)\cosh r}{(1+\lambda^2+(\lambda^2-1)\cosh r)^2}\,. \tag{5.2}$$

We want to check if this spacetime has closed timelike curves. The $(\phi-\phi)$-component of the metric is:

$$G_{\phi\phi} = k\frac{2\lambda^2(1+2H^2)\cosh r + (1-\lambda^2(1+2H^2))\cosh^2 r - (1+\lambda^2(1+2H^2))}{(1+\lambda^2+(\lambda^2-1)\cosh r)^2}\,. \tag{5.3}$$

We would like to know for which values of λ the numerator is always positive. The conclusion is:

$$\text{No CTC's} \leftrightarrow \lambda^2 \leq \frac{1}{1+2H^2}\,. \tag{5.4}$$

However, in this range, the background (5.1) becomes singular. The scalar curvature, eq. (5.2), blows up at:

$$r_{\text{sing}} = \operatorname{arccosh}\frac{\lambda^2+1}{1-\lambda^2}\,.$$

But the worldsheet conformal field theory is well-defined. This singularity corresponds to a ring of positive tension objects which are electrically charged under the NS-NS two form and charged under the gauge field. To have a more precise picture, we take for λ the limiting value that avoids closed timelike curves: $\lambda^2 = 1/(1+2H^2)$. Then we find that the singularity is located at:

$$r_{\text{sing}} = 2\rho_{\text{sing}} = 2\operatorname{arcsinh}\frac{1}{\sqrt{2}H}\,. \tag{5.5}$$

We observe that the locus of the annular singularity corresponds exactly to the critical radius above which closed timelike curves occurs, eq. (2.7). Furthermore the long strings become tachyonic in the Gödel/AdS_3 spacetime because they can wind around these closed timelike curves. Therefore, it is very tempting to interpret this annular singularity as a ring of condensed fundamental strings, protecting the causally unsafe region, in the same spirit as the supertubes domain walls considered in [7]. It would be interesting to make the connection with the D1/D5 black rings of [27].

5.2 Long string spectrum and chronology protection

Now we would like to know the expression of the long string spectrum as a function of the parameters of deformation H and λ. By turning on the gravitational deformation λ, the mass-shell condition (4.9) becomes (see [13]):

$$L_0 = \frac{s^2 + 1/4}{k+2} + \frac{k+2}{4} w_+^2 - w_+ \frac{E}{2} + \frac{1}{4(k+2)} \left(1 - \frac{\cos^2\zeta}{\lambda^2}\right) E^2 + h = 0 \tag{5.6}$$

and the energy will be given by:

$$E = (k+2) \left(1 - \frac{\cos^2\zeta}{\lambda^2}\right)^{-1} \left\{ w_+ \pm \sqrt{w_+^2 \frac{\cos^2\zeta}{\lambda^2} + \frac{4}{k+2} \left(\frac{\cos^2\zeta}{\lambda^2} - 1\right) \left(\frac{s^2 + 1/4}{k+2} + h\right)} \right\}. \tag{5.7}$$

so we find that there are no tachyonic behavior any longer in the spectrum, provided that

$$\lambda^2 \leq \cos^2\zeta\,.$$

Comparing with the relation (3.8), *this is exactly the same as the condition (5.4) required to avoid closed timelike curves!* Therefore, we see that there string theory offers a natural mechanism for chronology protection in Gödel-like universes.

6. Discussion and conclusions

We have studied a class of string backgrounds of the Gödel type, which are homogeneous and contain closed timelike curves. We have shown that these solutions are in fact exact conformal field theories obtained by a truly marginal deformation of the SL(2, $\mathbb{R}$) CFT. In heterotic superstrings, this deformation switches on a non-trivial gauge field in the bulk, and breaks all the supersymmetry. Embedded in type-II superstrings as a Kaluza-Klein reduction, this background preserves 8 supercharges coming from the right-movers, if we start with the $AdS_3 \times S^3 \times K3$ background describing NS5 branes and fundamental strings. We were able to construct the exact string spectrum for this solution, as well as the modular invariant partition function. We showed that the Gödel deformation preserves the unitarity of the physical spectrum, by proving a no-ghost theorem.

We then discussed the issue of long strings, which can wrap the closed timelike curves, first by showing that such classical solutions still exists in the deformed background. Then we studied the long string spectrum, expecting some pathology due to the closed timelike curves. We found that the spectrum of these long strings is highly tachyonic, for any nonzero value of the deformation parameter. We are aware that since there is no good definition of energy in this background, the notion of tachyon in this class of backgrounds is not completely well-defined. We expect that these long strings will destabilize the background. On the contrary, the spectrum of short strings is not pathologic, as we could infer because they are trapped in the center of the spacetime.

A possible resolution of the causality problem in this background has been proposed, by turning on another marginal deformation. It turned out that the closed timelike curves can be avoided, at the expense of going to a region of the moduli space where the background becomes singular. We observed that an annular singularity occurred precisely at the radius where the Gödel/AdS space started developing closed timelike curves, which were the source of instability for the long string probes. We showed that the spectrum of the resulting background is well-behaved. Therefore, we have interpreted this curvature, NS-NS two form and gauge field singularity as the result of the condensation of a ring of fundamental heterotic strings, in the same spirit as the supertubes of [7]. The proper description of this process would require a second-quantized framework — closed string field theory. If this is indeed the proper interpretation, it would be the first example of a stringy chronology protection in an exact string theory framework.

Since this background is continuously connected to $AdS_3 \times S^3 \times K3$ — the NS5/F1 background — we can expect some holographic interpretation. This issue have already been discussed in [5]. The authors showed that if one applies the covariant prescription of Bousso [28], on finds that the holographic screen, which is observer-dependent, encloses a *the causally safe region*; they argued that it leads to an "holographic protection" of these spacetimes. The radius of this cylindrical screen was shown to be:

$$r_{\text{screen}} = 2 \operatorname{arcsinh} \frac{1}{2H} . \tag{6.1}$$

This holographic screen is at finite distance, and the $H \to 0$ limit gives the usual holographic screen of AdS_3 at infinity. It is difficult to interpret the Gödel deformation of AdS_3 in terms of the boundary theory of AdS_3, and to make the connection with the aforementionned proposal. In fact, in AdS_3, the currents are non-normalizable operators and marginal current-current deformations corresponds to IR *irrelevant* deformations of the dual CFT. Such an observation has been made in [4] for another class of Gödel universe. Nevertheless, one might hope to understand the instability of the Gödel/AdS spacetime due to long string states as some instability in the boundary theory, triggered by some irrelevant operator, and then have an holographic picture of the bulk tachyon condensation. It have been shown [29] on a related example (BMPV black hole), that if we allow closed timelike curves in the bulk by increasing the rotation of the black hole, the states of the dual CFT will violate the unitarity bound. Since the spacetime we obtain after the condensation of the ring of fundamental strings seems stable, it would be really interesting to understand holography in this background, if any. We will address this problem in a future work.

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A. The $\mathrm{SL}(2,\mathbb{R})$ partition function

The computation of the one-loop vacuum amplitude has been done in [13], using a mixture of path integral and algebraic techniques. The result is, for the universal cover of $\mathrm{SL}(2,\mathbb{R})$

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at level $k+4$:

$$Z_{\mathrm{SL}(2,\mathbb{R})}(\tau) = \frac{4(k+4)\sqrt{k+2}}{\tau_2^{1/2}} \sum_{n,w,N,W} \int_0^1 ds_1 ds_2 dt_1 dt_2 \frac{e^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \times$$
$$\times\, e^{-\frac{(k+4)\pi}{\tau_2}|(s_1-t_1+w)\tau-(s_2-t_2+n)|^2} e^{\frac{(k+4)\pi}{\tau_2}|(t_1+W)\tau-(t_2+N)|^2} . \qquad \text{(A.1)}$$

The integration over the constraints s_1 and s_2 gives the spectra for all the flowed and unflowed representations (see [13] for details). The spectrum of primary states gives:

- *Discrete representations* representing strings trapped in the center of AdS_3; they appear in the range $\frac{1}{2} < \tilde{j} < \frac{k+3}{2}$. Their conformal weights are:

$$L_0 = -\frac{\tilde{j}(\tilde{j}-1)}{k+2} + w_+\left(-\tilde{m} - \frac{k+4}{4}w_+\right).$$

- *Continuous representations* with $\tilde{j} = -\frac{1}{2} + is$, $s \in \mathbb{R}_+$. They correspond to long strings that can reach the boundary of AdS_3. The spectral flow quantum number $w_+ = w + W$ represents their winding around the center near the boundary [23]. The weights are:

$$L_0 = \frac{s^2+1/4}{k+2} + w_+\left(-\tilde{m} - \frac{k+4}{4}w_+\right).$$

The relation between the unflowed and flowed eigenvalues follows from:

$$\tilde{J}^3_n = J^3_n - \frac{k+4}{2}w_+\delta_{n,0} ,$$

and the spacetime energy is given by:

$$E = \tilde{m} + \bar{\tilde{m}} + (k+4)w_+ = (k+4)(W+t_1) .$$

For latter convenience, it is useful to rewrite the $\mathrm{SL}(2,\mathbb{R})$ partition function as a twisted product of the coset $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ (the "cigar" 2d euclidean black hole) and a time-like $\mathrm{U}(1)$:

$$Z_{\mathrm{SL}(2,\mathbb{R})} = \frac{1}{\eta\bar{\eta}} \int d^2t\, Z_{\mathrm{cigar}}\begin{bmatrix} -t_1 \\ -t_2 \end{bmatrix} \times \qquad \text{(A.2)}$$
$$\times \sum_{N,W} q^{-\frac{1}{2}\left(\frac{N}{\sqrt{2(k+4)}} - \sqrt{\frac{k+4}{2}}(W+t_1)\right)^2} \bar{q}^{-\frac{1}{2}\left(\frac{N}{\sqrt{2(k+4)}} - \sqrt{\frac{k+4}{2}}(W+t_1)\right)^2} e^{-2i\pi N t_2} ,$$

where (see [36]):

$$Z_{\mathrm{cigar}}\begin{bmatrix} -t_1 \\ -t_2 \end{bmatrix} = 4\sqrt{(k+2)(k+4)} \sum_{m,w} \int d^2s\, \frac{\eta\bar{\eta}\, e^{\frac{2\pi}{\tau_2}(\mathrm{Im}(s_1\tau - s_2))^2}}{|\vartheta_1(s_1\tau - s_2|\tau)|^2} \times$$
$$\times\, e^{-\frac{(k+4)\pi}{\tau_2}|(s_1-t_1+w)\tau-(s_2-t_2+m)|^2} .$$

B. Coordinate systems and geodesics

In cylindrical coordinates, the metric of the AdS/Godel form is (see eq. (2.5):

$$ds^2 = dr^2 + \sinh^2 r d\alpha^2 - \left(d\tau + 4\Omega \sinh^2\left(\frac{r}{2}\right) d\alpha\right)^2 \qquad \text{with } 2\Omega = \sqrt{1+2H^2}\,.$$

One can go to cartesian coordinates by the coordinate transformation:

$$\begin{cases} e^{2x} = \cosh(2r) + \cos\left(\frac{\phi+\tau}{\Omega}\right)\sinh r \\ ye^{2x} = \frac{\Omega}{2}\sinh(2r)\sin\left(\frac{\phi+\tau}{\Omega}\right) \\ \tan\left(\frac{T+(\phi-\tau)/2}{\Omega}\right) = e^{-2r}\tan\left(\frac{\phi-\tau}{2\Omega}\right) \end{cases}$$

Then the metric reads:

$$ds^2 = dx^2 + \frac{1}{2}e^{2x}dy^2 - \left(dT + \sqrt{2}\Omega e^x dy\right)^2 .$$

The last useful coordinate system corresponds to the hyperbolic coordinates. It is obtained by:

$$x = -\ln X\,, \qquad y = \sqrt{2}Y\,.$$

Then one obtains:

$$ds^2 = \frac{dX^2 + dY^2 - \left(X^2 dT + 2\Omega dY\right)^2}{X^2}\,.$$

In these coordinates, the geodesic equation gives [12]:

$$\begin{aligned} \dot{T} &= (1+2H^2)E - \sqrt{1+2H^2}Xp_x \\ \dot{X} &= p_x X(Y - Y_0) \\ \dot{Y} &= -\sqrt{1+2H^2}EX + p_x X^2 \end{aligned}$$

where the energy E is the charge associated to the Killing vector ∂_T, and p_x the momentum associated to the Killing vector ∂_x.

C. No-ghost theorem for string theory on Gödel/AdS$_3$: sketch of a proof

Here we would like to outline the main steps of the no-ghost theorem for strings on AdS_3 and check whether it is still valid or not, in the case of the Gödel deformation. Unitarity for a theory living in a spacetime containing closed timelike curves is maybe not a well-defined concept. Nevertheless, we would like to know if, as in AdS_3, the Virasoro constraints are able to remove all the negative-norm states from the worldsheet conformal field theory. We will follow [30, 31, 32] and the appendix A of [23].[4] We consider string theory on (deformed) SL$(2,\mathbb{R})$ times a unitary "internal" CFT. The important observation is: writing the action for the deformed model as eq. (2.9), we see that, with a field redefinition, the actions of the deformed and the undeformed model are the same, *except that the structure of*

[4]It is also possible to use BRST methods, see [33] and [34].

the zero modes is different. The mixing in the zero modes have been given in eqs. (3.6) and (3.7). Note that the part of the CFT corresponding to the coset CFT $\mathrm{SL}(2,\mathbb{R})_k/\mathrm{U}(1)$ is left untouched. Here we consider for simplicity the case of bosonic strings, and the Gödel deformation is constructed as a Kaluza-Klein reduction along a coordinate $Y(z,\bar z)$ compactified at the fermionic point:

$$Q_y^2 = \frac{1}{2}\left(p - \frac{q}{2}\right)^2 , \qquad \bar Q_y^2 = \frac{1}{2}\left(p + \frac{q}{2}\right)^2 .$$

The extension to heterotic or type-II superstrings is straightforward. Since only the zero mode structure differs from the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{U}(1) \times CFT_{\text{internal}}$, given a representation of the zero modes, the structure of the Verma modules and the affine representations are the same, provided that no new null vectors appear. For convenience we will include the spectral flow quantum number in the parafermionic theory,[5] writing, for the undeformed $\mathrm{SL}(2,\mathbb{R})_k$:

$$\tilde L_0^{\mathrm{SL}(2,\mathbb{R})} = \tilde L_0^{\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)} - L_0^{(3)} \equiv \frac{c_2}{k-2} + \frac{1}{k}\left(m - \frac{k}{2}w_+\right)^2 - \frac{m^2}{k} \tag{C.1}$$

where c_2 is the second Casimir of the group, and:

$$L_n^{(3)} \equiv -\frac{1}{2k}\sum_m : J^3_m J^3_{n-m} : .$$

The proof of the no-ghost theorem can be carried out in three steps.

Step one: we define $\mathcal{F}$ as the subspace of the Hilbert space $\mathcal{H}$ consisting in states $|f\rangle \in \mathcal{F}$ such that:

$$L_n|f\rangle = J^3_n|f\rangle = 0 \qquad \text{for } n > 0 .$$

Then we have to show that the states of the form

$$L_{-n_1}\cdots L_{-n_N} J^3_{-m_1}\cdots J^3_{-m_M}|f\rangle \qquad \text{with } n_1 \geq \cdots \geq n_N ,\ m_1 \geq \cdots \geq m_M , \tag{C.2}$$

form a basis of $\mathcal{H}$. First, to prove that these states are linearly independent, we use an orthogonal decomposition of the Virasoro generators: $L_n = L_n^c - L_n^{(3)}$, and the fact that the states (C.2) are in one to one correspondence with the similar states constructed with the L_n^c's. The Virasoro algebra corresponding to the L_n^c's has a central charge $c^c = 25$, and therefore the associated Verma modules will contain no null states, provided that the conformal weights are strictly positive. The conformal weights for the "parafermionic" theory are, from eqns (3.6) and (3.7):

$$\begin{aligned} L_0^c &= L_0^{\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)} + \frac{1}{2}Q^2 + N_y + h_{\text{internal}}, \\ \bar L_0^c &= \bar L_0^{\mathrm{SL}(2,\mathbb{R})/\mathrm{U}(1)} + \frac{1}{2}\left(\bar Q\cos\zeta - \sqrt{\frac{2}{k}}\, m \sin\zeta\right)^2 + \bar N_y + \bar h_{\text{internal}}, \end{aligned}$$

[5] Usually the spectral flow is embedded in the $J^3\bar J^3$ lattice, by expressing the spectrum in terms of $\tilde m$ rather than in terms of m. Nevertheless, our expression of the spectrum of the coset looks like in [35] and [36].

and it is known that the coset theory $\mathrm{SL}(2,\mathbb{R})_k\,/\,\mathrm{U}(1)$ is unitary and the weights are strictly positive, provided that the spin of the discrete representations is restricted to: $0<j<k/2$. So the states (C.2) are linearly independent and form a basis of $\mathcal{H}$.

Step 2: we have to show that a physical state can be chosen such that it can be written as (C.2) with no L_{-n}.

Using the basis (C.2), we write any state $|\psi\rangle$ of $\mathcal{H}$ as:

$$|\psi\rangle = |sp\rangle + |\varphi\rangle\,,$$

where $|sp\rangle$ is a spurious state, i.e. with some n_i's non zero, and $|\varphi\rangle$ has all its n_i's being zero. Then we use the fact that, for a critical string background ($c=26$), if we act on a on-shell (i.e. $L_0=1$) spurious state with $L_{n>0}$, we obtain again a spurious state. So we can map every physical state to a state $|\phi\rangle$ such that $L_{n>0}|\phi\rangle=0$.

Step 3: we have to show that if a physical state $|\varphi\rangle$ can be written as (C.2) with no L_{-n}, then $J^3_{n>0}|\phi\rangle=0$. First, this implies that

$$L^{(3)}_{n>0}|\varphi\rangle = 0\,. \tag{C.3}$$

We want to show that the states (C.3) are such that: $J^3_{n>0}|\varphi\rangle=0$. It will be true if there are no null states in the Virasoro descendents of $L^{(3)}$ for the left-movers, and $\bar{L}^{(3)}$ for the right-movers. The weights of this $c=1$ conformal field theory are:

$$\begin{aligned} L^{(3)}_0 &= -\frac{1}{k}\left(m\ \cos\zeta + \sqrt{\frac{k}{2}}\,\sin\zeta\,\bar{Q}\right)^2 \\ \bar{L}^{(3)}_0 &= -\frac{1}{k}\bar{m}^2\,. \end{aligned}$$

So, this statement would be true if $L^{(3)}_0\neq 0$ and $\bar{L}^{(3)}_0\neq 0$ for all on-shell states. What remains to do is to examine separately the states with $J^{(3)}_0=0$ or $\bar{J}^{(3)}_0=0$.

- For the right algebra, the mass-shell condition (4.7) for $\bar{m}=0$ reads:

  $$\bar{L}_0 = \frac{c_2}{k-2} + \frac{k}{4}w^2 + \bar{N}_{\text{coset}} + \frac{\cos\zeta}{2}\bar{Q}^2 + \bar{N}_3 + \bar{h}_{\text{internal}} = 1$$

  where $\bar{N}_{\text{coset}}$ is the grade of the state with respect to the currents $\bar{J}^{\pm}$ and $\bar{N}_3$ is the grade w.r.t. the current $\bar{J}^3$. So, since $\bar{h}_{\text{internal}}>0$ there are no on-shell states obtained by the action of $\bar{J}^3_{-n}$ (i.e. with $\bar{N}_3\neq 0$) for the continuous representations. It is also true for the discrete representations, provided that $0<\tilde{j}<k/2$, as in the undeformed $\mathrm{SL}(2,\mathbb{R})$.

- For the left algebra, we are looking for states with

  $$m\cos\zeta + \sqrt{\frac{k}{2}}\,\sin\zeta\,\bar{Q} = 0\,. \tag{C.4}$$

We have the corresponding on-shell condition:

$$L_0 = \frac{c_2}{k-2} + N_{\rm coset} + \frac{1}{k}\left(m - \frac{k}{2}w_+\right)^2 + N_3 + \frac{1}{2}Q^2 + N + h_{\rm internal} = 1\,.$$

In a representation $\mathcal{D}^+_{\tilde{j}}$, we have $\tilde{m} = \tilde{j} + q$, $q \in \mathbb{Z}$, and, if q is negative:

$$-q \leq N_{\rm coset}. \tag{C.5}$$

So we have for the weights of the coset part:

$$\begin{aligned} L_0^{\rm coset} &= \frac{-\tilde{j}(\tilde{j}-1)}{k-2} + \frac{\tilde{m}^2}{k} + N_{\rm coset} \\ &= \frac{q^2}{k} + \frac{\tilde{j}}{k-2}\left[1 - \frac{2\tilde{j}}{k}\right] + \frac{2}{k}\tilde{j}q + N_{\rm coset} \end{aligned}$$

If $\tilde{j}$ is taken in the range $0 < \tilde{j} < k/2$, then with the inequality (C.5) we obtain that $L_0^{\rm coset} > 0$. So there are no on-shell states with $N_3 \geq 1$.

This completes the proof of the no-ghost theorem.

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